Integer Points in Plane Regions and Exponential Sums

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How many integer points (m, n) lie inside a large circle, or in the annulus between two circles? There are approaches by real-variable approximation theory, or by Fourier analysis. The same ideas occur in both, and the latest results use both methods at different stages of the argument.

Analytic number theory is about counting the number of sets of integers satisfying certain conditions. There are two famous questions.

1. The number of prime numbers in a short interval.
2. The number of integer points inside a circle: (m, n) with

$$m^2 + n^2 \le R^2.$$

In question 1, prime numbers are defined negatively by ruling out multiples of smaller prime numbers. One asks: how many solutions of

$$pq = N + h,$$

with p a prime number in some range, and h a small integer? This is a very hard problem. An easier version is to change from prime numbers to square-free numbers, defined negatively by not being multiples of the squares of prime numbers. Now the relation is

$$p^2 q = N + h, \quad p^2 q \simeq N.$$

A version with two terms,

$$p_1^2 q_1 - p_2^2 q_2 = h, \quad \frac{p_1}{p_2} \simeq \sqrt{\frac{q_2}{q_1}},$$

leads to an approximate equation involving rational numbers.

The two questions can be generalised to two problems.

Problem A Count the number of integer points inside a closed curve (as an asymptotic formula).

Problem B Count the number of integer points (m, n) close to an arc of a curve $y = f(x)$, so that

$$|n - f(m)| \le \delta. \tag{1}$$

Upper or lower bounds are still interesting when there is no asymptotic formula.

Variants of these problems have integer points (m, n) replaced by rational points, either projectively (points $(\frac{m}{q}, \frac{n}{q})$ with $q \le Q$ corresponding to an integer point (m, n, q) in projective space) or generally (points $(\frac{m}{n}, \frac{r}{q})$ with $n \le N, q \le Q$).

The tools of analytic number theory are:

1. *Discreteness*. An integer n with $|n| < 1$ is zero.
2. *Cauchy's inequality*. Mean squares are positive.
3. *Fourier analysis*, usually Poisson summation.

Problem A by Fourier analysis leads to exponential sums.

Problem C Estimate the sum

$$\sum_{M}^{M'} e(f(m)), \quad e(x) = \exp 2\pi i x, \quad M' \leq 2M. \tag{2}$$

or the two-variable sum

$$\sum_{h=H}^{H'} \sum_{m=M}^{M'} e(hf(m)), \quad H' \leq 2H, \quad M' \leq 2M. \tag{3}$$

To compare the problems we suppose that $M \leq m \leq 2M$, with $f(m) = TF(\frac{m}{M})$ in Problems A and C (T has the dimensions of area), and $f(m) = NF(\frac{m}{M})$ in Problem B (N has the dimensions of length, so MN corresponds to T in Problem A). Here $F(x)$ is a bounded function defined on an interval containing [1,2]. Certain derivatives and determinants of derivatives will occur in the estimates. These derivatives are assumed to be bounded in modulus.

Standard case. Any derivative in the denominator is bounded away from zero.

Non-standard case. Some derivative which should be in the denominator becomes very small.

In a non-standard case some quantity is approximately constant. If its value has a good rational approximation a/q with q small, then strange things happen, and the argument may fail. If there is no good rational approximation, then the argument can be modified.

Here are some typical functions $F(x)$.

Problem A $F(x) = \sqrt{1 - x^2}$ (Gauss's circle problem), $F(x) = 1/x$ (Dirichlet's divisor problem).

Problem B $F(x) = 1/x$ (prime numbers), $F(x) = 1/\sqrt{x}$ (square-free numbers).

Problem C As in Problems A and B, and also $F(x) = \log x$ (size of the Riemann zeta-function).

The first non-trivial results in each problem typically show a saving of $1/3$ on some exponent in the trivial bound, leading to exponents with denominators 3 or 6 in the final result.

Circle Problem (Sierpinski 1906). The number of integer solutions of $x^2 + y^2 \leq R^2$ is

$$\pi R^2 + O(R^{2/3}).$$

Divisor problem (integer points inside an hyperbola) (Voronoi 1903). The number of positive integer solutions of $mn \le T$ is

$$T \log T + (2\gamma - 1)T + O(T^{1/3}(\log T)^c).$$

Size of the zeta function (Hardy and Littlewood 1921).

$$\zeta\left(\frac{1}{2} + it\right) = O(T^{1/6} \log T).$$

Integer points close to curves (Vinogradov and van der Corput independently about 1914–18).

$$R = 2\delta M + O((MN)^{1/3}(\log MN)^c).$$

Rational points close to a curve (Huxley 1994a). With (m, n) in (1) replaced by $(m/q, n/q)$ with highest common factor $(m, n, q) = 1$ and $q \le Q$, $Mq \le m \le 2Mq$, and δ replaced by δ/Q, for $F''(x) \ne 0$

$$R = O\left(\delta M Q^2 \left(\frac{MN}{\delta}\right)^{\epsilon} + (MN)^{1/3} Q\right).$$

General case (Huxley 2000). With (m, n) in (1) replaced by $(m/n, r/q)$ with highest common factors $(m, n) = (r, q) = 1$, $n \le M$, $q \le Q$ and δ replaced by δ/Q^2, with $f(x) = TF(x)$, satisfying $F'(x)$, $F''(x)$ and $3F''(x)^2 - 2F'(x)F^{(3)}(x)$ all non-zero,

$$R = O(\delta^{1/4} M^2 + (M^2 Q^2 T(\delta M^2 + 1))^{1/3}(MQ^2T)^{\epsilon}).$$

In these results c is some fixed exponent, ϵ is an exponent which can be taken arbitrarily small. The constant factor suppressed in the $O(.)$ notation is constructed from the range of values of the derivatives of $F(x)$, and from ϵ where present.

Van der Corput (1920 and later papers) developed an iterative procedure for estimating exponential sums.

Step A First differences (or mean square over short intervals). This step keeps M the same, and replaces $F(x)$ by $F'(x)$ (to a first approximation).

Step B Poisson summation. This step is used if $M > \sqrt{T}$ and $F''(x)$ is non-zero. It keeps T the same, reduces M, and replaces $F(x)$ by the complementary function $G(y)$ satisfying $F'(G'(y)) = y$.

The iteration continues until M is small enough for the sum to be estimated trivially, or until the new function that replaces $F(x)$ becomes non-standard. The B step must be followed by an A step, but A steps can be iterated, so the iteration has a branched tree structure. Each A step introduces an extra variable, summed over a short range. A more complicated version of the iteration allows A and B steps with respect to these subsidiary variables, or in several variables at once. In practice the iteration is stopped after a few steps, either because error terms from a B step in one variable become large when summed over the other variables, or because large parts of the transformed sums are non-standard. Van der Corput's iteration can now be easily studied in Graham and Kolesnik (1991).

For exponential sums with $(\log M)/(\log T)$ very small, Vinogradov's mean value method using high moments gives a better estimate than repeated A steps. Vinogradov's mean value method is a single step that does not lead on to further iteration. In some ways it is analogous to taking r-th differences.

Huxley (1989) discovered an analogous iteration that gives upper bounds for Problem B, the number of integer points close to a curve.

Step A Differencing using the interpolation polynomial over short intervals. This step keeps M the same, reduces N, increases δ, and replaces $F(x)$ by $F^{(r)}(x)$ (to a first approximation), where r is the order of differencing.

Step B Interchange x and y. This step is used if $M > N$ and $F'(x)$ is non-zero. It interchanges M and N, reducing M and increasing δ. The function $F(x)$ is replaced by its inverse function.

The iteration ends with a trivial estimate, which is $O(M)$ if $\delta \le 1/2$, $O(\delta M)$ if the previous step has made $\delta > 1/2$. The use of r-th differences makes this iteration more powerful than the van der Corput iteration for exponential sums. Recent work (Filaseta and Trifonov 1996, Huxley and Sargos 1995) uses essentially an A step followed by a B step. The B step is elaborated by focussing on short arcs of the curve. Sargos introduced the classification into major and minor arcs. A major arc is a region where there is a good rational approximation $y = g(x)$ (with small denominator) to the equation of the curve, such that any integer point (m, n) close to this arc of the curve must satisfy $n = g(m)$. Other regions of the curve are called minor arcs. Huxley and Sargos (1995) gave a simple upper bound in Problem B.

$$R = O\left(\Delta^{\frac{2}{r(r+1)}} M + \delta^{\frac{2}{r(r-1)}} M + \left(\frac{\delta}{\Delta}\right)^{\frac{1}{r}} + 1\right), \tag{4}$$

provided that $F^{(r)}(x)$ is non-zero and

$$\Delta = T/M^r < 1.$$

The first two terms on the right of (4) are the estimate from the minor arcs. The third term on the right of (4) is the possible contribution of a single major arc, and the term 1 covers trivial cases. An object of current research (Filaseta and Trifonov 1996, Huxley and Sargos 2000) is to reduce the first term in (4) under further conditions, such as the non-vanishing of $F^{(r-1)}(x)$ as well.

Lower bounds can be given in Problem B if δ is not too small (Huxley 1996b). The number of integer points found in this way is less than the expected number $2\delta M$, and they all lie on major arcs. When $(\log M)/(\log N)$ is large, the rational approximations $y = g(x)$ on the major arcs are constructed by an iterative process involving differencing. At present the denominator of approximation has to be a power of two for the iteration to proceed for more than two steps. This restriction permits an approximation argument in the 2-adic metric, and avoids looking in a short interval for a solution of a congruence to an arbitrary modulus q. The estimate that starts the iteration is

$$R \geq c \min\left(\delta M, \frac{\delta^4 M^3}{N}\right) \text{ when } \quad \delta \geq \frac{d}{\sqrt{M}}, \tag{5}$$

when $N \geq M$ and $F''(x)$ is non-zero and numerically less than 1. Here c and d are explicit positive constants constructed from the range of values of $F''(x)$. The iteration leads to similar but weaker lower bounds when the second derivative $F''(x)$ is replaced by $F^{(r)}(x)$ in the condition.

The van der Corput iteration for exponential sums shows that, for large δ, R is not only non-zero, but approximately equal to $2\delta M$. The lower bounds are still positive for δ a little smaller than this range.

For both problems B and C there is a deeper method in the middle range, where $(\log M)/(\log N)$ is near 1 for points close to a curve, or $(\log M)/(\log T)$ is near $1/2$ for exponential sums. For points close to a curve, Swinnerton Dyer's method uses a fourth moment short interval estimate on the minor arcs (Swinnerton-Dyer (1974) considered points on the curve, with $M = N$ and $\delta = 0$, so he had no major arcs). The bounds take the form

$$R = O((MN)^{3/10}(\log\ MN)^{1/10} + \text{terms in } \delta) \tag{6}$$

for $F''(x)$ and $F^{(3)}(x)$ non-zero, and $N \geq M$. For the latest version, see Huxley (1999). Similar arguments with higher moments come up against determinants which do not factor completely into linear factors.

For exponential sums the deeper method is that of Bombieri and Iwaniec (1986), which uses high moments of short exponential sums. Each short sum is labelled by a rational approximation a/q to $\frac{1}{2}f''(x)$, and is transformed by Poisson summation (incorporating the finite Fourier transform modulo q). On major arcs q is small, and the transformed sum can be estimated at once. The transformed minor arc sums are estimated in mean 2k-th power ($k = 4$ in Bombieri and Iwaniec (1986) and Huxley and Watt (1988), $k = 5$ in Watt (1989) and subsequent papers). After some simplification, the large sieve inequality is used to bound a bilinear form

$$\sum_h \sum_j \alpha_h \beta_j e(\mathbf{x}^{(h)} \cdot \mathbf{y}^{(j)}),$$

where h is the integer variable introduced by Poisson summation, and j indexes the rationals a_j/q_j that label the short sums on the minor arcs. The vectors $\mathbf{x}^{(h)}$ lie in a box $\mathbf{A}$ in four-dimensional space, and the vectors $\mathbf{y}^{(j)}$ lie in a box $\mathbf{B}$. The upper bound for the bilinear form requires an estimate for the sum

$$\sum_h \sum_i |\alpha_h \alpha_i|$$

taken over a neighbourhood of the diagonal in $\mathbf{A} \times \mathbf{A}$. The size of the neighbourhood is determined by the size of the box $\mathbf{B}$. It is sufficient to count the number of pairs of vectors

$\mathbf{x}^{(h)}, \mathbf{x}^{(i)}$ in the given neighbourhood. This is the First Spacing Problem; the Second Spacing Problem is the analogue for the vectors $\mathbf{y}^{(j)}$.

The First Spacing Problem involves counting the number of sets of integers $h_1, \ldots, h_{2k}$ with

$$h_1^i + \cdots + h_k^i \simeq h_{k+1}^i + \cdots + h_{2k}^i$$

simultaneously for $i = 2, 3/2, 1, 1/2$. Vinogradov's mean value method produces conditions like this, but with integer exponents and exact equality. Progress with the First Spacing Problem was made in Watt (1989) and in Huxley and Kolesnik (1991).

In the Second Spacing Problem, a pair of vectors $\mathbf{y}^{(j)}, \mathbf{y}^{(k)}$ in a neighbourhood of the diagonal in $\mathbf{B} \times \mathbf{B}$ corresponds to a relation between two minor arcs which we call resonance. The four coordinates give four coincidence conditions, whose strengths are measured by real numbers $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, each less than unity. The first coincidence condition implies the existence of the 'magic matrix' M, which has small integer entries and determinant one, with

$$\begin{pmatrix} a_k \\ q_k \end{pmatrix} = M \begin{pmatrix} a_j \\ q_j \end{pmatrix}.$$

The factor Δ_1 has to be saved in the construction in order to get even a trivial estimate.

When the magic matrix is fixed, then the second coincidence condition holds on a block of consecutive minor arcs, the domain of the magic matrix. The factor Δ_2 saved leads to the non-trivial results of Bombieri and Iwaniec (1986), Huxley and Watt (1988), Watt (1989) and Huxley and Kolesnik (1991).

The next step is to assume the first and second coincidence conditions (the first, alas, with borrowed strength which must be repaid). Among the minor arcs in the domain of the magic matrix, those on which the third and fourth coincidence conditions also hold, they give rise to integer points close to a certain curve, the resonance curve. More precisely, a resonance curve corresponds to a block of U consecutive minor arcs, with $U = O(1/\Delta_4)$. The length of the resonance curve grows like $U^{3/2}$. The result of Huxley (1993) was obtained by choosing U so that the resonance curve had bounded length, and thus a bounded number of integer points close to it. This gives a saving of a factor $\Delta_4^{2/3}$ from the third and fourth coincidence conditions together.

Work in progress (Huxley 2001a, 2001b) uses a bound of the form (6). There is a possible saving by a factor $\Delta_4^{7/10} < \Delta_4^{2/3}$. However the typical resonance curve has a cusp, with the gradient on one branch tending to infinity, and on the other branch tending to zero. At the cusp and the ends of the resonance curve, the conditions for (6) do not hold, and a weaker argument must be found.

Further improvements would come if different resonance curves could be compared, even by showing that most resonance curves do not have an integer point very close to the cusp.

The usual test of exponential sum bounds is the size of the Riemann zeta function

$$\zeta\left(\tfrac{1}{2} + iT\right) = O(T^{\theta}(\log T)^{c}).$$

Hardy and Littlewood (1921) had $\theta = 1/6$. Rankin (1955) and Graham and Kolesnik (1991) calculated the limit of van der Corput's iteration in one variable as $\theta = 0.164\ldots$. Using the subsidiary variables in van der Corput's iteration gives exponents which are smaller, but which have $\theta > 0.1618\ldots$ (Graham and Kolesnik 1991). Bombieri and Iwaniec (1986) obtained $\theta = 0.1607\ldots$, and the latest form of their method (Huxley 1993a) has $\theta = 0.156140\ldots$. Huxley (2001a) uses (6) to reach $\theta = 0.156098\ldots$. The best possible estimates in the First and Second Spacing Problems in the Bombieri-Iwaniec method would reach $\theta = 3/20$.

The Bombieri-Iwaniec method works best when the ratio $(\log M)/(\log T)$ is close to 1/2. Sargos (1995) has a modified form of the method which works well near 2/5. Kolesnik has a different construction of resonance curves suited to the case when all magic matrices are upper triangular, which happens when the ratio is below 3/7. Huxley and Kolesnik made extensive calculations of an iteration based on the Kolesnik resonance curve. Their results are summarised in chapter 19.3 of Huxley (1996a).

There seems to be no analogue of the Bombieri-Iwaniec method to exponential sums in two variables involving a function $F(x, y)$, because simultaneous Diophantine approximations to three second order partial derivatives with the same denominator cannot be as accurate as an approximation to one second derivative for a function of one variable. The form of the large sieve inequality allows a small saving where the second variable is used as a parameter that modifies the sum. A similar argument allows a coefficient $a(m)$ to be inserted in the sums (2), where $a(m)$ has some integer period $q < M$, and it can be used to extend the estimates from the zeta function to the Dirichlet L-functions (Huxley and Watt 1997).

The best treatment for exponential sums with $(\log M)/(\log T)$ not near 1/2 is to apply some A steps of van der Corput's iteration (or a B step followed by some A steps) until the new parameters M and T are in the Bombieri-Iwaniec range. The extra variables from the A steps give parameters that modify the main sum, so there is a small extra saving.

The Bombieri-Iwaniec method does work if there is one function, but two integer variables occurring as $hf(m)$ as in (3) (Iwaniec and Mozzochi 1989) or as $f(m+h)-f(m-h)$ (Heath-Brown and Huxley 1990). The First Spacing Problem is different in these constructions, but the Second Spacing Problem is almost the same; the main difference is that $f(x)$ in (3) corresponds to $f'(x)$ in (2). This leads to an estimate for the generalised circle problem. Let C be a closed convex plane curve of area A for which the arc length s is three times continuously differentiable with respect to the tangent angle ψ (the circle is the case $s = \psi$, $A = \pi$). Let D be a plane domain bounded by a copy of C enlarged by a factor $M \geq 2$. Then the number of integer points (m, n) in D is

$$AM^2 + O(M^\theta (\log\ M)^c)$$

with $\theta = 46/73 = 0.6301\ldots$; the constant of proportionality depends on C but not on the orientation of the domain D (Huxley 1993b). In the Dirichlet divisor problem, the number of positive integer solutions of $mn \leq M^2$ is

$$2M^2 \log\ M + (2\gamma - 1)M^2 + O(M^\theta (\log\ M)^c)$$

with the same θ. The possible improvement in the Second Spacing Problem would give $\theta = 0.6298\ldots$. The analogous sums with $f(m+h) - f(m-h)$ in place of $hf(m)$ in (3) lead to short interval mean square bounds for exponential sums (Heath-Brown and Huxley 1990, Huxley 1994b).

For exponential sums in one variable, the form of the underlying function $F(x)$ does not matter very much. In two variables the non-standard regions of the double sum depend on the shape of $F(x, y)$, for instance on whether the second order terms are elliptic or hyperbolic. The functions $F(x)$ which arise from the famous problems satisfy algebraic differential equations. For problem B, points close to a curve, it is sometimes possible to use this. The relation

$$\frac{p_1}{p_2} \simeq \sqrt{\frac{q_2}{q_1}}$$

arises in the study of square-free numbers in a short interval. The curve $F(x) = 1/\sqrt{x}$ has a major arc around the point (1,1) with an approximation $y = g(x)$, an explicit Pade approximant rational function. This idea was introduced by Roth (1951). Filaseta and Trifonov combined this with a differencing argument to discuss the gaps between square-free numbers. Let $s_1, \ldots, s_N$ be the square-free numbers in the range $1, \ldots, M$, so that

$$N \simeq \frac{6M}{\pi^2}.$$

Then the maximum gap satisfies

$$s_{n+1} - s_n = O(M^{1/5}(\log\ M)^c)$$

(Filaseta and Trifonov 1992), and Erdös's asymptotic formula

$$\sum_{1}^{N-1} (s_{n+1} - s_n)^{\gamma} \simeq \beta(\gamma) M,$$

where $\beta(\gamma)$ is a constant determined by the exponent γ, holds for $0 \leq \gamma < 59/16 = 3.6875$ (Huxley 1999). These arguments use the upper bounds in Problem B.

For the case $\delta = 0$ of Problem B, points actually on the curve, Bombieri and Pila (1989) have upper bounds using intersection theory in algebraic geometry, which are essentially best possible for algebraic curves. In general the constant of proportionality in the upper bound is shown to exist by a compactness argument, and it cannot be calculated explicitly or uniformly, which rules out some applications. For results of the quality of (6), the curve must be differentiable a large finite number of times. It seems to be possible to extend Bombieri and Pila's argument to non-zero δ, but the upper bound increases rapidly as δ moves away from zero.

The lower bounds in Huxley (1996b) imply but do not improve the famous result of Bambah and Chowla (1947) that the gap between numbers less than N which are sums of two squares is $O(N^{1/4})$.

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